Global nonexistence of solutions for the viscoelastic wave equation of Kirchhoff type with high energy

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Abstract: In this paper we consider the viscoelastic wave equation of Kirchhoff type:

$$u_{tt} - M(\|\nabla u\|_2^2)\Delta u + \int_0^t g(t-s)\Delta u(s)\mathrm{d}s + u_t = |u|^{p-1}u$$

with Dirichlet boundary conditions. Under some suitable assumptions on g and the initial data, we established a global nonexistence result for certain solutions with arbitrarily high energy.

Keywords: global nonexistence; Kirchhoff type; viscoelastic wave equation; high energy.

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1 Introduction

In this paper we consider the following problem

$$\begin{cases} u_{tt} - M(\|\nabla u\|_2^2)\Delta u + \int_0^t g(t-s)\Delta u(s)\mathrm{d}s + |u_t|^{m-1}u_t = |u|^{p-1}u, & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times [0,\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$, $p > 1$, $M(s)$ is a nonnegative C^1 function like $M(s) = a + bs^\gamma$ for $s \geq 0$, $a \geq 0, b \geq 0, a+b > 0, \gamma > 0$ and $g(t)$ represents the kernel of memory term.

Problem (1.1) without the viscoelastic term ($i.e., g = 0$) has been extensively studied and many results concerning global existence, decay and blow-up have been established. For example, the following equation

$$u_{tt} - M(\|\nabla u\|_2^2)\Delta u + g(u_t) = f(u), \quad (x,t) \in \Omega \times (0,\infty), \tag{1.2}$$

has been considered by Matsuyama and Ikehata in [10], for $g(u_t) = \delta|u_t|^{p-1}u_t$ and $f(u) = \mu|u|^{q-1}u$. The authors proved existence of the global solutions by using Faedo-Galerkin method and the decay of energy based on the method of Nakao [13]–[15]. Later, Ono [16] investigated equation (1.2) for $M(s) = bs^\gamma$ and $f(u) = |u|^{p-2}u$. When $g(u_t) = -\Delta u_t$, u_t or $|u_t|^\beta u_t$, the author showed that the solutions blow up in finite time with $E(0) \leq 0$. For $M(s) = a+bs^\gamma$ and $g(u_t) = u_t$, this model was considered by the same author in [17]. By applying the potential well method he obtained the blow-up properties with positive initial energy $E(0)$. Recently, Zeng et al. [27]

studied equation (1.2) for the case $g(u_t) = u_t$ with initial condition and zero Dirichlet boundary condition. By using the concavity argument, they proved that the solutions to equation (1.2) blow up in finite time with arbitrarily high energy.

In the case of $M \equiv 1$ and in the presence of the viscoelastic term ($i.e., g \neq 0$), the equation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\mathrm{d}s + |u_t|^{m-1}u_t = |u|^{p-1}u, \quad (x,t) \in \Omega \times (0,\infty), \tag{1.3}$$

was studied by Messaoudi in [11], where the author proved that any weak solution with negative initial energy blows up in finite time if $p > m$ and

$$\int_0^\infty g(s)\mathrm{d}s \leq \frac{p-1}{p-1+1/(p+1)},$$

while the solution continue to exist globally for any initial data in the appropriate space if $m \geq p$. This blow-up result was improved by the same author in [12] for positive initial energy under suitable conditions on g, m and p. More recently, Wang [22] investigated equation (1.3) and established a blow-up result with arbitrary positive initial energy. In the related work, Cavalcanti et al. [1] studied the following equation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\mathrm{d}s + a(x)u_t + |u|^\gamma u = 0, \quad (x,t) \in \Omega \times (0,\infty), \tag{1.4}$$

where $a : \Omega \longrightarrow R^+$ is a function which may be null on a part of Ω. Under the condition that $a(x) \geq a_0 > 0$ on $\omega \subset \Omega$, with ω satisfying some geometric restrictions and $-\xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t)$, $t \geq 0$ to guarantee $\|g\|_{L^1((0,\infty))}$ is small enough, they proved an exponential decay rate.

When $g \neq 0$ and M is not a constant function, problems related to (1.1) have been treated by many authors. Wu and Tsai [24] considered the global existence, asymptotic behavior and blow-up properties for the following equation

$$u_{tt} - M(\|\nabla u\|_2^2)\Delta u + \int_0^t g(t-s)\Delta u(s)\mathrm{d}s - \Delta u_t = f(u), \quad (x,t) \in \Omega \times (0,\infty), \tag{1.5}$$

with the same initial and boundary conditions as that of problem (1.1). To obtain the decay result, they assumed that the nonnegative kernel $g'(t) \leq -rg(t), \forall\ t \geq 0$ for some $r > 0$. In [23], Wu then extended the result of [24] under a weaker condition on g (i.e., $g'(t) \leq 0$ for $t \geq 0$). For other papers related to existence, uniform decay and blow-up of solutions of nonlinear wave equations, see [2, 3, 5, 6, 7, 8, 9, 18, 19, 20, 21, 25, 26] and references therein.

Motivated by the above research, we consider problem (1.1) for $m = 1$ in this paper and establish a global nonexistence result for certain solutions with arbitrarily high energy. In this way, we can extend the result of [27] to nonzero term g and the result of [22] to nonconstant $M(s)$. We also obtain the new result for blow-up properties of local solution with arbitrarily high energy. Throughout the rest of this paper, we always assume that $m = 1$.

The structure of this paper is as follows. In section 2, we present some assumptions, notations and main result. Section 3 is devoted to the proof of the main result.

2 Preliminaries and main result

In this section, we shall give some assumptions, notations and main result. We first give the following assumptions:

(A1) $g \in C^1([0,\infty))$ is a non-negative and non-increasing function satisfying

$$1 - \int_0^\infty g(s)\mathrm{d}s = l > 0.$$

(A2) The function $e^{\frac{t}{2}}g(t)$ is of positive type in the following sense:

$$\int_0^t v(s) \int_0^s e^{\frac{s-z}{2}} g(s-z)v(z)\mathrm{d}z\mathrm{d}s \geq 0, \quad \forall\ v \in C^1([0,\infty)) \quad \text{and} \quad \forall\ t > 0.$$

In order to prove our result, we make the following assumption on M and g :

(A3) There exist two positive constants, m_1 and α, such that

$$\frac{p+1}{2}\overline{M}(s) - \left[M(s) + \frac{p+1}{2}\int_0^t g(\tau)\mathrm{d}\tau\right] s \geq m_1 s^\alpha, \quad \forall\ s \geq 0,$$

where $\overline{M}(s) = \displaystyle\int_0^s M(\tau)\mathrm{d}\tau$.

Remark 1 *It is clear that when $M(s) = a + bs^\gamma$ for $s \geq 0$, $a \geq 0$, $b \geq 0$, $a + b > 0$, $\gamma > 0$ and $p > 1 + 2\gamma$, condition (A3) can be replaced by*

$$\int_0^\infty g(\tau)\mathrm{d}\tau < \begin{cases} \frac{p-1}{p+1}a, & \text{if} \quad a > 0 \quad \text{and} \quad b \geq 0, \\ \frac{(p-1-2\gamma)b}{C_p^\gamma(p+1)(\gamma+1)}\|u_0\|_2^{2\gamma}, & \text{if} \quad a = 0 \quad \text{and} \quad b > 0, \end{cases} \tag{2.1}$$

which is the same as the one in [22, Theorem 1.1] for the case $a = 1$ and $b = 0$, where C_p is the constant from the Poincaré inequality $\|u(t)\|_2^2 \leq C_p\|\nabla u(t)\|_2^2$.

Indeed, by straightforward calculation, we obtain

$$\begin{aligned} &\frac{p+1}{2}\overline{M}(s) - \left[M(s) + \frac{p+1}{2}\int_0^t g(\tau)\mathrm{d}\tau\right] s \\ =&\frac{p+1}{2}\left(as + \frac{b}{\gamma+1}s^{\gamma+1}\right) - as - bs^{\gamma+1} - \frac{(p+1)s}{2}\int_0^t g(\tau)\mathrm{d}\tau \\ =&\frac{p-1}{2}as + \frac{(p-1-2\gamma)b}{2(\gamma+1)}s^{\gamma+1} - \frac{(p+1)s}{2}\int_0^t g(\tau)\mathrm{d}\tau. \end{aligned}$$

If $a > 0$ and $b \geq 0$, it follows from (2.1) that $\int_0^\infty g(\tau)\mathrm{d}\tau < \frac{p-1}{p+1}a$. Thus, we have

$$\begin{aligned} &\frac{p+1}{2}\overline{M}(s) - \left[M(s) + \frac{p+1}{2}\int_0^t g(\tau)\mathrm{d}\tau\right] s \\ >&\frac{p-1}{2}as + \frac{(p-1-2\gamma)b}{2(\gamma+1)}s^{\gamma+1} - \frac{(p+1)s}{2}\left[\frac{p-1}{p+1}a - \frac{\frac{p-1}{p+1}a - \int_0^\infty g(\tau)\mathrm{d}\tau}{2}\right] \\ =&\frac{(p-1-2\gamma)b}{2(\gamma+1)}s^{\gamma+1} + \frac{1}{2}\left[\frac{p-1}{p+1}a - \int_0^\infty g(\tau)\mathrm{d}\tau\right] s \geq \frac{1}{2}\left[\frac{p-1}{p+1}a - \int_0^\infty g(\tau)\mathrm{d}\tau\right] s. \end{aligned}$$

Therefore, we can choose $m_1 = \frac{1}{2}\left[\frac{p-1}{p+1}a - \int_0^\infty g(\tau)\mathrm{d}\tau\right]$ and $\alpha = 1$ in condition (A3).

If $a=0$ and $b>0$, then

$$
\begin{aligned}
&\frac{p+1}{2}\overline{M}(s)-\left[M(s)+\frac{p+1}{2}\int_0^t g(\tau)\mathrm{d}\tau\right]s=\frac{(p-1-2\gamma)b}{2(\gamma+1)}s^{\gamma+1}-\frac{(p+1)s}{2}\int_0^t g(\tau)\mathrm{d}\tau\\
>&\frac{(p-1-2\gamma)b}{2(\gamma+1)}s^{\gamma+1}-\frac{(p+1)s}{2}\left[\frac{(p-1-2\gamma)b}{C_p^\gamma(p+1)(\gamma+1)}\|u_0\|_2^{2\gamma}-\frac{\frac{(p-1-2\gamma)b}{C_p^\gamma(p+1)(\gamma+1)}\|u_0\|_2^{2\gamma}-\int_0^\infty g(\tau)\mathrm{d}\tau}{2}\right]\\
=&\frac{(p-1-2\gamma)b}{2(\gamma+1)}s\left(s^\gamma-\frac{1}{C_p^\gamma}\|u_0\|_2^{2\gamma}\right)+\frac{(p+1)s}{4}\left[\frac{(p-1-2\gamma)b}{C_p^\gamma(p+1)(\gamma+1)}\|u_0\|_2^{2\gamma}-\int_0^\infty g(\tau)\mathrm{d}\tau\right].
\end{aligned}
$$

Taking $s=\|\nabla u(t)\|_2^2$, applying Lemma 3.3 below and Poincaré's inequality, we can get

$$
\begin{aligned}
&\frac{p+1}{2}\overline{M}(\|\nabla u(t)\|_2^2)-\left[M(\|\nabla u(t)\|_2^2)+\frac{p+1}{2}\int_0^t g(\tau)\mathrm{d}\tau\right]\|\nabla u(t)\|_2^2\\
>&\frac{(p-1-2\gamma)b}{2(\gamma+1)}\|\nabla u(t)\|_2^2\left(\|\nabla u(t)\|_2^{2\gamma}-\frac{1}{C_p^\gamma}\|u_0\|_2^{2\gamma}\right)\\
&+\frac{p+1}{4}\|\nabla u(t)\|_2^2\left[\frac{(p-1-2\gamma)b}{C_p^\gamma(p+1)(\gamma+1)}\|u_0\|_2^{2\gamma}-\int_0^\infty g(\tau)\mathrm{d}\tau\right]\\
\geq&\frac{(p-1-2\gamma)b}{2(\gamma+1)}\|\nabla u(t)\|_2^2\left(\|\nabla u(t)\|_2^{2\gamma}-\frac{1}{C_p^\gamma}\|u(t)\|_2^{2\gamma}\right)\\
&+\frac{p+1}{4}\|\nabla u(t)\|_2^2\left[\frac{(p-1-2\gamma)b}{C_p^\gamma(p+1)(\gamma+1)}\|u_0\|_2^{2\gamma}-\int_0^\infty g(\tau)\mathrm{d}\tau\right]\\
\geq&\frac{(p-1-2\gamma)b}{2(\gamma+1)}\|\nabla u(t)\|_2^2\left(\|\nabla u(t)\|_2^{2\gamma}-\|\nabla u(t)\|_2^{2\gamma}\right)\\
&+\frac{p+1}{4}\|\nabla u(t)\|_2^2\left[\frac{(p-1-2\gamma)b}{C_p^\gamma(p+1)(\gamma+1)}\|u_0\|_2^{2\gamma}-\int_0^\infty g(\tau)\mathrm{d}\tau\right]\\
=&\frac{p+1}{4}\left[\frac{(p-1-2\gamma)b}{C_p^\gamma(p+1)(\gamma+1)}\|u_0\|_2^{2\gamma}-\int_0^\infty g(\tau)\mathrm{d}\tau\right]\|\nabla u(t)\|_2^2.
\end{aligned}
$$

So, we can choose $m_1=\frac{p+1}{4}\left[\frac{(p-1-2\gamma)b}{C_p^\gamma(p+1)(\gamma+1)}\|u_0\|_2^{2\gamma}-\int_0^\infty g(\tau)\mathrm{d}\tau\right]$ and $\alpha=1$ in condition (A3).

Next, we introduce some notations. The energy functional $E(t)$ and an auxiliary functional $I(u)$ of the solution $u(t)$ of problem (1.1) are defined as follows:

$$
E(t):=E(u(t))=\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\overline{M}(\|\nabla u\|_2^2)-\frac{1}{2}\int_0^t g(s)\mathrm{d}s\|\nabla u\|_2^2+\frac{1}{2}(g\circ\nabla u)(t)-\frac{1}{p+1}\|u\|_{p+1}^{p+1}, \tag{2.2}
$$

and

$$
I(u)=M(\|\nabla u\|_2^2)\|\nabla u\|_2^2-\|u\|_{p+1}^{p+1}, \tag{2.3}
$$

where

$$
(g\circ w)(t)=\int_0^t g(t-s)\|w(t,\cdot)-w(s,\cdot)\|_2^2\mathrm{d}s.
$$

As in [22, 27], we can get

$$
\frac{d}{\mathrm{d}t}E(t)=-\|u_t\|_2^2-\frac{1}{2}g(t)\|\nabla u\|_2^2+\frac{1}{2}(g'\circ\nabla u)(t)\leq 0, \tag{2.4}
$$

for $t \geq 0$. Then we have

$$E(t) = E(0) - \int_0^t \|u_s\|_2^2 \mathrm{d}s + \frac{1}{2}\int_0^t (g' \circ \nabla u)(s)\mathrm{d}s - \frac{1}{2}\int_0^t g(s)\|\nabla u(s)\|_2^2 \mathrm{d}s. \tag{2.5}$$

Now we are in a position to state our main result.

Theorem 2.1 *Assume that M and g satisfy assumptions (A1)-(A3). Suppose further that $1 < p \leq \frac{n}{n-2}$ when $n \geq 3$, $1 < p < \infty$ when $n = 1, 2$. Let u be a solution of problem (1.1) with initial data $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $u_1 \in H_0^1(\Omega)$ satisfying*

$$E(0) > 0, \tag{2.6}$$

$$I(u_0) < 0, \tag{2.7}$$

$$\int_\Omega u_0 u_1 \mathrm{d}x > 0, \tag{2.8}$$

$$\|u_0\|_2^2 > C_p \left(\frac{p+1}{m_1} E(0)\right)^{1/\alpha}. \tag{2.9}$$

Then the solution of problem (1.1) blows up in finite time $0 < T^* < +\infty$, which means that

$$\lim_{t \to T^{*-}} \left(\|u(t)\|_2^2 + \int_0^t \|u(s)\|_2^2 \mathrm{d}s \right) = \infty, \tag{2.10}$$

where C_p is a constant from the Poincaré inequality and m_1 comes from condition (A3).

3 Proof of main result

Before we start to prove Theorem 2.1, it is necessary to state the local existence theorem for problem (1.1), whose proof follows the arguments in [19, 24].

Theorem 3.1 *Assume that (A1) holds, and $1 < p \leq \frac{n}{n-2}$ when $n \geq 3$, $1 < p < \infty$ when $n = 1, 2$. For $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $u_1 \in H_0^1(\Omega)$ and $M(\|\nabla u_0\|_2^2) > 0$, problem (1.1) has a unique local solution*

$$u \in C([0,T]; H_0^1(\Omega) \cap H^2(\Omega)), \quad u_t \in C([0,T]; L^2(\Omega)) \cap L^2([0,T]; H_0^1(\Omega))$$

for the maximum existence time $T > 0$.

The proof of Theorem 2.1 relies on the following lemmas.

Lemma 3.2 ([22, Lemma 2.1]) *Assume that $g(t)$ satisfies assumptions (A1)-(A2), and $H(t)$ is a function which is twice continuously differentiable satisfying*

$$\begin{cases} H''(t) + H'(t) > \int_0^t g(t-s) \int_\Omega \nabla u(s) \nabla u(t) dx ds \\ H(0) > 0, \quad H'(0) > 0, \end{cases} \tag{3.1}$$

for every $t \in [0, T)$, where $u(t)$ is the corresponding solution of problem (1.1) with u_0 and u_1. Then the function $H(t)$ is strictly increasing on $[0, T)$.

Lemma 3.3 *Suppose that* $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ *and* $u_1 \in H_0^1(\Omega)$ *satisfy*

$$\int_\Omega u_0 u_1 \mathrm{d}x > 0. \tag{3.2}$$

If the solution $u(t)$ *of problem* (1.1) *exists on* $[0, T)$ *and satisfies*

$$I(u(t)) < 0, \tag{3.3}$$

then $\|u(t)\|_2^2$ *is strictly increasing on* $[0, T)$.

Proof. Since $u(t)$ is the solution of problem (1.1), by a simple computation, we have

$$\begin{aligned}
\frac{1}{2}\frac{d^2}{\mathrm{d}t^2}\int_\Omega |u(x,t)|^2 \mathrm{d}x = & \int_\Omega (|u_t|^2 + uu_{tt})\mathrm{d}x \\
= & \|u_t\|_2^2 - M(\|\nabla u\|_2^2)\|\nabla u\|_2^2 + \|u\|_{p+1}^{p+1} \\
& + \int_0^t g(t-s)\int_\Omega \nabla u(s)\nabla u(t)\mathrm{d}x\mathrm{d}s - \int_\Omega uu_t \mathrm{d}x \\
> & - \int_\Omega uu_t \mathrm{d}x + \int_0^t g(t-s)\int_\Omega \nabla u(s)\nabla u(t)\mathrm{d}x\mathrm{d}s,
\end{aligned}$$

where the last inequality is derived by (3.3). Then we get

$$\frac{d^2}{\mathrm{d}t^2}\int_\Omega |u(x,t)|^2 \mathrm{d}x + \frac{d}{\mathrm{d}t}\int_\Omega |u(x,t)|^2 \mathrm{d}x > \int_0^t g(t-s)\int_\Omega \nabla u(s)\nabla u(t)\mathrm{d}x\mathrm{d}s.$$

Therefore, by using Lemma 3.2, we finish our proof. $\square$

Lemma 3.4 *If* $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ *and* $u_1 \in H_0^1(\Omega)$ *satisfy the assumptions in Theorem 2.1, then the solution* $u(t)$ *of problem* (1.1) *satisfies*

$$I(u(t)) < 0, \tag{3.4}$$

$$\|u\|_2^2 > C_p\left(\frac{p+1}{m_1}E(0)\right)^{1/\alpha}, \tag{3.5}$$

for all $t \in [0, T)$.

Proof. We will prove the above lemma by contradiction. First we assume that (3.4) is not true over $[0, T)$, it means that there exists a time t_0 such that

$$t_0 = \min\{t \in (0, T) : I(u(t)) = 0\}. \tag{3.6}$$

Since $I(u(t)) < 0$ on $[0, t_0)$, by Lemma 3.3, we see that $\int_\Omega u^2 \mathrm{d}x$ is strictly increasing over $[0, t_0)$, which implies

$$\int_\Omega u^2 \mathrm{d}x > \int_\Omega u_0^2 \mathrm{d}x > C_p\left(\frac{p+1}{m_1}E(0)\right)^{1/\alpha}. \tag{3.7}$$

And by the continuity of $\int_\Omega u^2 \mathrm{d}x$ on t, we note that

$$\int_\Omega u^2(t_0)\mathrm{d}x > C_p \left(\frac{p+1}{m_1}E(0)\right)^{1/\alpha}. \tag{3.8}$$

On the other hand, by (2.2) and (2.5), we get

$$\overline{M}(\|\nabla u(t_0)\|_2^2) - \int_0^{t_0} g(s)\mathrm{d}s\|\nabla u(t_0)\|_2^2 + (g \circ \nabla u)(t_0) - \frac{2}{p+1}\|u(t_0)\|_{p+1}^{p+1} \leq 2E(0). \tag{3.9}$$

Combining (3.9) with (3.6) yields

$$\begin{aligned} &\frac{p+1}{2}\overline{M}(\|\nabla u(t_0)\|_2^2) - \frac{p+1}{2}\int_0^{t_0} g(s)\mathrm{d}s\|\nabla u(t_0)\|_2^2 \\ &\quad + \frac{p+1}{2}(g \circ \nabla u)(t_0) - M(\|\nabla u(t_0)\|_2^2)\|\nabla u(t_0)\|_2^2 \leq (p+1)E(0). \end{aligned} \tag{3.10}$$

By (A3), we get

$$m_1\|\nabla u(t_0)\|_2^{2\alpha} < (p+1)E(0). \tag{3.11}$$

i.e.,

$$\|\nabla u(t_0)\|_2^2 < \left(\frac{p+1}{m_1}E(0)\right)^{1/\alpha}. \tag{3.12}$$

By Poincaré's inequality, we have

$$\|u(t_0)\|_2^2 < C_p \left(\frac{p+1}{m_1}E(0)\right)^{1/\alpha}. \tag{3.13}$$

Obviously, there is a contradiction between (3.8) and (3.13), thus we prove that

$$I(u(t)) < 0, \tag{3.14}$$

for every $t \in (0, T)$. By Lemma 3.3, it follows that $\int_\Omega u^2 \mathrm{d}x$ is strictly increasing on $[0, T)$, which implies that

$$\int_\Omega u^2 \mathrm{d}x \geq \int_\Omega u_0^2 \mathrm{d}x > C_p \left(\frac{p+1}{m_1}E(0)\right)^{1/\alpha}, \tag{3.15}$$

for every $t \in [0, T)$. This completes the proof of Lemma 3.4 . □

Lemma 3.5 ([4]) Assume that $P(t) \in C^2$, $P(t) \geq 0$, satisfies the inequality

$$P(t)P''(t) - (1+\theta)P'^2(t) \geq 0,$$

for certain real number $\theta > 0$, and $P(0) > 0$, $P'(0) > 0$. Then there exists a real number T^* such that $0 < T^* \leq P(0)/\theta P'(0)$ and

$$P(t) \to \infty$$

as $t \to T^{*-}$.

Proof of Theorem 2.1. We prove our main result by adopting concavity method, and define an auxiliary function by

$$G(t) = \|u(t)\|_2^2 + \int_0^t \|u(s)\|_2^2 \mathrm{d}s + (T_0 - t)\|u_0\|_2^2 + \beta(t_2 + t)^2, \tag{3.16}$$

where T_0, t_2 and β are positive constants, which will be chosen later.

A straightforward calculation gives

$$\begin{aligned} G'(t) &= 2\int_\Omega uu_t \mathrm{d}x + \|u(t)\|_2^2 - \|u_0\|_2^2 + 2\beta(t_2 + t) \\ &= 2\int_\Omega uu_t \mathrm{d}x + 2\int_0^t (u(s), u_s(s))\mathrm{d}s + 2\beta(t_2 + t), \end{aligned} \tag{3.17}$$

consequently,

$$\begin{aligned} G''(t) =& 2\int_\Omega |u_t|^2 \mathrm{d}x + 2\int_\Omega uu_{tt}\mathrm{d}x + 2\int_\Omega uu_t \mathrm{d}x + 2\beta \\ =& 2\|u_t\|_2^2 - 2M(\|\nabla u\|_2^2)\|\nabla u\|_2^2 + 2\int_0^t g(t-s)\int_\Omega \nabla u(s)\nabla u(t)\mathrm{d}x\mathrm{d}s \\ & - 2\int_\Omega uu_t \mathrm{d}x + 2\int_\Omega uu_t\mathrm{d}x + 2\|u\|_{p+1}^{p+1} + 2\beta \\ =& 2\|u_t\|_2^2 - 2M(\|\nabla u\|_2^2)\|\nabla u\|_2^2 + 2\|u\|_{p+1}^{p+1} + 2\int_0^t g(t-s)\mathrm{d}s\|\nabla u\|_2^2 \\ & + 2\int_0^t g(t-s)\int_\Omega \nabla u(t)(\nabla u(s) - \nabla u(t))\mathrm{d}x\mathrm{d}s + 2\beta. \end{aligned} \tag{3.18}$$

We will use the following Young inequality to estimate the fifth term of the right hand side of (3.18),

$$rs \leq \frac{r^2}{2\epsilon} + \frac{\epsilon s^2}{2}$$

where $\epsilon = \frac{1}{2}$, $r \geq 0$ and $s \geq 0$. We obtain

$$\int_0^t g(t-s)\int_\Omega |\nabla u(t)||\nabla u(s) - \nabla u(t)|\mathrm{d}x\mathrm{d}s \leq \int_0^t g(s)\mathrm{d}s\|\nabla u(t)\|_2^2 + \frac{1}{4}(g\circ\nabla u)(t), \tag{3.19}$$

Substitute (2.2) and (3.19) for the third and the fifth terms of the right hand side of (3.18), respectively, we have

$$\begin{aligned} G''(t) \geq& (p+3)\|u_t\|_2^2 + (p+1)\overline{M}(\|\nabla u\|_2^2) - 2M(\|\nabla u\|_2^2)\|\nabla u\|_2^2 - (p+1)\int_0^t g(s)\mathrm{d}s\|\nabla u\|_2^2 \\ & - 2(p+1)E(t) + \left(p + \frac{1}{2}\right)(g\circ\nabla u)(t) + 2\beta. \end{aligned} \tag{3.20}$$

By (A3), we deduce

$$G''(t) > (p+3)\|u_t\|_2^2 + 2m_1\|\nabla u\|_2^{2\alpha} - 2(p+1)E(t) + \left(p+\frac{1}{2}\right)(g\circ\nabla u)(t) + 2\beta. \tag{3.21}$$

Noting that (2.5), we obtain that

$$-E(t) \geq -E(0) + \int_0^t \|u_s\|_2^2 \mathrm{d}s. \tag{3.22}$$

Combining (3.21)-(3.22) yields

$$\begin{aligned} G''(t) >&(p+3)\|u_t\|_2^2 + 2m_1\|\nabla u\|_2^{2\alpha} - 2(p+1)E(0) + \left(p+\frac{1}{2}\right)(g\circ\nabla u)(t) \\ &+ 2(p+1)\int_0^t \|u_s\|_2^2 \mathrm{d}s + 2\beta, \end{aligned} \tag{3.23}$$

by (3.5), we see that

$$2m_1\|\nabla u\|_2^{2\alpha} - 2(p+1)E(0) + \left(p+\frac{1}{2}\right)(g\circ\nabla u)(t) > 0.$$

which means that $G''(t) > 0$ for every $t \in (0,T)$. Thus, by $G'(0) > 0$ and $G(0) > 0$, we get $G'(t)$ and $G(t)$ are strictly increasing on $[0,T)$.

Thus, we can let β satisfy

$$(p+1)\beta < 2m_1\|\nabla u\|_2^{2\alpha} - 2(p+1)E(0) + (p+\frac{1}{2})(g\circ\nabla u)(t).$$

Moreover, we let T_0 and t_2 satisfy that

$$T_0 \geq \frac{4}{p-1}\frac{G(0)}{G'(0)},$$

$$\frac{p-1}{2}\left(\int_\Omega u_0u_1\mathrm{d}x + \beta t_2\right) \geq \|u_0\|_2^2.$$

Letting

$$\begin{aligned} A :=&\|u(t)\|_2^2 + \int_0^t \|u(s)\|_2^2\mathrm{d}s + \beta(t_2+t)^2, \\ B :=&\frac{1}{2}G'(t), \\ C :=&\|u_t(t)\|_2^2 + \int_0^t \|u_s(s)\|_2^2\mathrm{d}s + \beta. \end{aligned}$$

Since we have assumed that the solution $u(t)$ to problem (1.1) exists for every $t \in [0,T)$, where T is sufficiently large, we have

$$\begin{aligned} G(t) &\geq A, \\ G''(t) &\geq (p+3)C \end{aligned}$$

for every $t \in [0,T_0)$. Then it follows that

$$G''(t)G(t) - \frac{p+3}{4}(G'(t))^2 \geq (p+3)(AC-B^2).$$

Furthermore, we have

$$Ar^2 - 2Br + C = \int_\Omega (ru(t) - u_t(t))^2 \mathrm{d}x + \int_0^t \|ru(s) - u_s(s)\|_2^2 \mathrm{d}s + \beta[r(t_2 + t) - 1]^2 \geq 0,$$

for every $r \in \mathbb{R}$, which implies that $B^2 - AC \leq 0$.

Thus, we obtain

$$G''(t)G(t) - \frac{p+3}{4}(G'(t))^2 \geq 0,$$

for every $t \in [0, T)$.

As $\frac{p+3}{4} > 1$, letting $\theta = \frac{p-1}{4}$, according to Lemma 3.5, there exists a real number T^* such that $T^* < G(0)/\theta G'(0) \leq T_0$ and we have

$$\lim_{t \to T^{*-}} G(t) = \infty,$$

i.e.,

$$\lim_{t \to T^{*-}} \left(\|u(t)\|_2^2 + \int_0^t \|u(s)\|_2^2 \mathrm{d}s \right) = \infty. \tag{3.24}$$

This completes the proof of Theorem 2.1. □

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